# SKETCH OF THE ORIGIN AND DEVELOPMENT OF GEOMETRY PRIOR TO 1850.[1]

[CONTINUED.]

THE intellectual movement which went out from Italy in the period of the Renaissance continued, as already said, under the direction first of France and then of England and Germany. But toward the end of the eighteenth century, after Euler had ceased to "calculate and to live"[2] and Lagrange (1736–1813) had pitched his tent in France, this country put itself once more at the head of the mathematical world. Not only by Clairaut, D'Alembert[3] (1716–1783), Condorcet (1743–1794),[4] Laplace (1749–1827), Legendre (1752–1833), Ampère (1775-1836),[5] Poinsot (1777–1859), Poisson (1781–1840),[6] and other lesser men, did this country direct the study of higher analysis and mathematical physics, but it also led scientists back to the study of the geometric forms in the sense—*mutatis mutandis*—in which that was understood by the learned men of old. This was accomplished by the work of Monge,[7] Carnot[8] (1753–1823) —disciple of Monge at the school of engineering of Mézières—and Poncelet[6] (1788–1867), the splendid triad on whose memorable pro-

[1] Translated from the Italian of Loria by Dr. George Bruce Halsted.

[2] Expression used by Condorcet in his *Éloge de M. Euler*.

[3] J. Bertrand, *D'Alembert* (Paris, 1889).

[4] Arago, *op. cit.*, 2, 1854. [5] Arago, *loc. cit.* [6] Arago, *loc. cit.*

[7] See Charles Dupin, *Essai historique sur les services et les travaux scientifiques de Gaspard Monge* (Paris, 1819); Arago, *loc. cit.*; K. Fink, "Monge" (*Korresp.-Bl. f. d. Gel. und Realschule*, Tübingen, 1892, pp. 263–289, 339–359).

[8] Arago, *op. cit.*, 1, 1854; K. Fink, *Lazare Nicolas Marguerite Carnot. Sein Leben und seine Werke nach den Quellen dargestellt* (Tübingen, 1894).

[9] See E. Holst, *On Poncelet's Betyding for Geometrien* (Christiania, 1878). Bertrand, *Éloges académiques* (Paris, 1890).

ductions we must pause an instant. The first, by uniting in one body of doctrine the few rules of perspective which the architects and artists had created for themselves, impelled by needs which they felt while cultivating their arts, and filling up with felicitous genius the many serious gaps which their ensemble presented, united his own name with a new branch of geometry, *descriptive geometry*.[1] Here is not the place to summarise the first known treatise of such branch of mathematics; we only note that if its publication had a practical and almost national motive,[2] Monge also attributed to the new science a theoretical value derived from the facility with which it enables one to conceive and then to study geometrical figures; moreover he was careful to compare his processes with those proper to analysis, showing their essential identity. By the classical works just cited and still more by the incomparable lectures at the polytechnic school of Paris,[3] of which he was one of the first and most splendid ornaments,[4] he put anew in honor the

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[1] Whoever desires further particulars relating to the prehistory and history of descriptive geometry is referred to the first section of Vol. I. (Leipzig, 1884) of the *Lehrbuch der darstellenden Geometrie* of Chr. Wiener (1826–1896). Here let it suffice to mention *La théorie et la practique de la coupe des pierres et des bois ou Traité de stéréotomie* (1738–1739) of Frézier (1682–1773).

[2] In fact the *Programme* prefixed to it commences with the words: "Pour tirer la nation française de la dépendance où elle a été jusqu'à présent de l'industrie étrangère etc." (Monge, *Géometrie descriptive*, Paris, an VII.)

[3] See Arago, *op. cit.*, 3, 1855, p. 70 and following; Jacobi, "Ueber die Pariser polytechnische Schule" (*Ges. Werke*, 7, Berlin, 1891, pp. 355–370).

[4] Monge had for collaborators in his work of reform several colleagues—among others Lacroix (1765–1843) and Hachette (1769–1834)—and many scholars. Among these it is but just to make here special mention of Charles Dupin (1764–1873) who "above the others like an eagle soars," thanks specially to his two works *Développements de géométrie* (Paris, 1813) and *Applications de géométrie et de mécanique* (Paris, 1822) of which we shall have occasion to speak further on; meanwhile we here mention the eulogy written on him by Bertrand (*Éloges academiques*, Paris, 1890) and the article "Dupin" by K. Fink (1851–1898) (*Korresp.-Bl. f. d. Gel. und Realsch.*, 1893, pp. 1–27). Nor can we omit making honorable mention also of the *Mémoire sur les lignes du second ordre* (Paris, 1817) and of the *Application de la théorie des transversales* (Paris, 1818) of Brianchon (1783–1864), writings of which the first contains the fundamentals of a synthetic theory of conic sections, with special regard to their construction, founded in part on the theorem of Pascal and its correlative (which Brianchon himself made known in the XIII cah. of the *Journ. Éc. Pol.* and afterwards generalised in the Vol. IV. of the *Ann. de Math.*), in part on anharmonic ratio; while the second aims to show of what

study of geometry founded on the direct consideration of the figures which are its object, and facilitating the conception of geometrical figures of three dimensions, he prepared for the application of stereometric considerations to the research of the properties of plane figures, which Pappus had only vaguely seen (see § 5) and which represents to-day one of the most fertile methods of investigation and demonstration that geometry boasts of.[1]

By the side of the *Géométrie descriptive* of Monge must be placed the *Géométrie de position* (Paris, 1803) of Carnot, the latter having as object in common with the former to give geometry that generality of concepts and methods and in consequence that clear and disinvolved procedure which were believed to belong exclusively to algebra. The reader who knows only the title of the most important geometric work of the "organiser of victory" might believe that its subject coincides with that unfolded by the modern works on projective geometry. Nothing would be more inexact; the intention which Carnot set himself is to determine the mutual dependence which exists between the different aspects which may be assumed by a figure satisfying certain conditions in correspondence with the various positions of which the data are susceptible, a study enabling us to dispense with that minute enumeration of

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fecundity for constructions are the theorems of Menelaus and Ceva, an aim analogous to what Servois (1767–1847) (comp. J. Boyer, *Le mathématicien Franc-Courtois François Joseph Servois*, Besançon, 1895) contemplated in composing his *Solutions peu connues de différents problèmes de géométrie pratique* (Paris, 1805).

[1] The influence of Monge was not limited to France and the neighboring countries; this is proved by the writings of Garbinski (1796–1848) and Sapalski (1791–1838), viz., the *Synthetic Exposition of the Properties of Ruled Surfaces* (Warsaw, 1822) of the former and the excellent treatise on descriptive geometry (Warsaw, 1822 and 1849) of the latter. Moreover such influence was not quickly spent, since it may still be recognised in the methods of teaching elementary geometry; this is proved by the attempt to do away with the old rigorous separation between planimetry and stereometry made for the first time by Bretschneider (1808–1878) in 1844 (*Lehrgebäude der niederen Geometrie*), repeated later by Méray (*Nouveaux éléments de géométrie*, Paris, 1875), and successfully accomplished in 1884 by De Paolis in his excellent *Elementi di geometria* (Torino). With respect to the influence of Monge on the recent development of graphic methods see H. T. Eddy, *Modern Graphical Developments* (Math. Papers read at Chicago Congress, New York, 1896).

the various cases of the figure, to which the ancient geometers were constrained. Such aim is to-day attained by a path much more smooth and clear, that is by the methodical use of the signs in geometry; for this reason the *Géométrie de position* presents to-day little more than historical interest, and must be mentioned less on account of the ideas set forth, than because of the applications there given, among which we mention here the principles of polygonometry and of polyhedrometry, the theory of barycenters and the theory of transversals, a conspicuous part of which is the celebrated proposition which is to-day designated as Carnot's theorem.[1] It is well also to notice here that the memoir *De la corrélation des figures de la géométrie* (Paris, 1801) is a first sketch of the *Géométrie de position*, while the *Lettre du citoyen Carnot au citoyen Bossut* (Paris, 1799) and the *Mémoirs sur la relation qui existe entre les distances respectives de cinq points quelconques pris dans l'espace* (Paris, 1806) have with it numerous points of contact; that further the *Réflexions sur la métaphysique du calcul infinitésimal* (Paris, 1796) form an integral part of the *Géométrie de position*, as some suppose, is something that cannot be accepted without mature examination and engthy discussion.

The writings of Monge and Carnot prepared effectively for the revival of pure geometry; this must be dated from the appearance (1822) of the *Traité des propriétés projectives des figures* of Poncelet.[2] To convince the reader how memorable is this date, it is sufficient to notice that it is in Poncelet's great work that is demonstrated the power of central projection and of the principle of continuity as instruments of research and auxiliaries in demonstration;[3] that

[1] We believe it opportune to note also that Carnot uses the word *equipollence* in a sense analogous to that in which much later it was used by Bellavitis.

[2] Only in passing we mention the fierce war waged on this work when just issued, which was in a great part due to Poncelet's ideas not having been understood.

[3] Whether the ancients knew how to employ perspective in theoretical research is a question which we cannot here examine (see Chasles, *Aperçu historique*, 2d ed., p. 74, note); the analogous question concerning the principle of continuity is studied by C. Taylor in the note "On the History of Geometrical Continuity" (*Cambridge Proc.*, 1880 and 1881) and in the preface (pp. LXXIII. seq.) to the work *Ancient and Modern Geometry of Conics*, where, among others, are de-

in it the deeper study of homology in the plane and in space serves, not only as a prelude, but further as most effective preparation, for the study of univocal correspondence between two systems of equal number of dimensions (see Chapter VIII.); that there also the old notions concerning polarity[1] with respect to a conic[2] and those discoveries of Monge's school relating to a quadric surface prepare the law of reciprocity which, recognised by Vieta,[3] by Philip van Lansberg (1561–1632),[4] and by Snellius (1581–1626)[5] in the geometry of the sphere, was destined to be enunciated in all its generality and called by the name of *principle of duality*, four years later (see "Considérations philosophiques sur les éléments de la science de l'étendue," *Ann. de Math.*, 16, 1825–1826,)[6] by Gergonne, (1781–1859.[7]) While an importance rather transitory pertained to the considerations set forth by Poncelet concerning the ideal chords of conic sections,—on which one relating the history of the theory of imaginaries in geometry might pause,—on the other hand great value must be attributed to the *Supplément* of the work in question, from which, among other things, we learn of the existence of four cones in any pencil of quadrics; moreover the investigations concerning polygons inscribed in one conic and circumscribed about

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scribed the efforts of Boscovich (1711–1787) to give a firm basis to the afore-mentioned principle.

[1] The name *pole* was proposed by Servois ("Solution du premier, etc." *Ann. de Math.*, T. I., 1810–1811), that of *polar* by Gergonne ("Théorie analytique des poles, des lignes, et des surfaces du second ordre," *Ibid.*, T. III., 1812–1813).

[2] A special application of polarity is met with in the *Traité analytique des sectiones coniques* (Paris, 1707, p. 275), of the Marquis de l'Hôpital (1661–1704). See also the "Rectifications."

[3] *Variorum de rebus mathematicis responsorum liber octavus* (Tours, 1593).

[4] See K. Fink, *Kurzer Abriss der Geschichte der Elementar-Mathematik* (Tübingen, 1890), p. 196.

[5] *Doctrina triangolorum canonica* (Leyden, 1627).

[6] *Ann. de Math.*, 19, 1827, where one finds for the first time the expression *class of a curve*.

[7] Further particulars on the origin of duality are found in the *Vorlesungen über continuirliche Gruppen* by Lie, edited by G. Scheffers (Leipzig, 1893), a work in which is also set forth a most important generalisation of the dualistic relation. See moreover *Julius Plückers gesammelte wissenschaftliche Abhandlungen*, I. (Leipzig, 1895), p. 619.

another possess permanent value; nay, if the place assigned to a mathematical research should be determined, not so much on the basis of the intrinsic value which it seems to possess, as rather by its originality and the novelty and variety of the investigations which branch out from it, it follows that to the last cited theme belongs a position second to a very few. And on this account and because it is extremely instructive to contemplate all the aspects revealed by modern mathematics in a single and minor subject, we wish to pause an instant on the so-called *polygons* of Poncelet.[1]

13. The problem: "Given in a plane two conic sections, to construct a polygon of a certain number of sides, which shall be inscribed in one of the given curves and circumscribed about the other," seems determinate, and from the year 1817 Poncelet reckoned[2] it among those to which could be applied the methods he was then elaborating; however when we remember that, if the two conics are circles the data must satisfy a certain relation[3] in order that the problem be possible, the suspicion arises that in the general case something analogous takes place. And in fact five years later Poncelet demonstrated that the problem enunciated does not admit, generally speaking, of a solution, but that when it has one, there is an infinite number (*Traité des propr. projectives*, §§ 530 et seq.; see *Applications* cited, 1, Paris, 1862, p. 348 et seq.). This memorable discovery did not pass unnoticed; which is proved by the fact that not more than six years after the time when it became public, Jacobi (1804–1851)[4] in the course of his notable researches

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[1] For greater particulars refer to the "Note historique, critique et philosophique," inserted by Poncelet in the *Applications d'analyse et de géométrie, etc.* I. (Paris, 1862), to the lecture of the author, entitled *I poligoni di Poncelet* (Turin, 1889), and to the complement published in the *Bibliotheca mathematica*, 1889, with the title "Rassegna di alcuni scritti sui poligoni di Poncelet."

[2] "Réflexions sur l'usage de l'analyse algébrique en géométrie," inserted first in the *Ann. de Math.*, 8, 1817, and then in the *Applications d'analyse et de géométrie*, 2 (Paris, 1864), p. 466.

[3] In the case of a triangle this was discovered by Euler (*Nov. Comm. Petrop.*, 2, 1751) and for the other polygons it was investigated by his disciple Nicholas Fuss (1755–1826) (*Nova Acta Petrop.*, 10, 1794, pub. 1797, and 13, 1798, pub. 1827) and found in part by Steiner (*Journ. f. Math.*, 2, 1827).

[4] Lejeune-Dirichlet, "Gedächtnissrede auf C. G. J. Jacobi" (*Berl. Abh.*, 1852).

concerning the theory of elliptical functions observed—in the celebrated paper "Ueber die Anwendung der elliptischen Transcendenten auf ein bekanntes Problem der Elementar-Geometrie" (*Journ. f. Math.*, 3, 1828)—how this theory was able to furnish a complete general solution (nay, the only explicit one, that is to say, other than by a recurring way, which is known even to-day) of the problem which consists in seeking the relations existing between the radii and the distance of the centers of two circumferences which admit a polygon of Poncelet of a given number $n$ of sides. And since, as Laplace thought, "discoveries consist in bringing together ideas capable of uniting and which were before separate," we must without hesitation give to Jacobi's observation the name of discovery. Nor need we wonder therefore at perceiving how the writing just cited was judged by Legendre worthy to be abridged in one of the supplements to the third edition of his voluminous *Théorie des fonctions elliptiques et des intégrales eulériennes*, and that Richelot (1808–1875) thought proper to devote two memoirs[1] to develop and complete the ideas of his own master and also to show how these could be applied to solve the analogous problems of spherical geometry.

At the end of the above mentioned memoir Jacobi noted that through the theory of elliptic functions it would have been most interesting to institute for two conics considerations analogous to those explained by him for two circumferences, even declaring himself disposed to return himself to the subject; but later, probably attracted by more important problems, he did not execute his project, which for nearly forty years waited in vain for some one to execute it; finally in 1865 Rosanes and Pasch gave in the notable memoir "Ueber das einem Kegelschnitte umbeschriebene und einem anderen einbeschriebene Polygon" (*Journ. f. Math.*, 64) a transcendental solution of the problem, which consists in the in-

[1] "Anwendung der elliptischen Transcendenten auf die sphärischen Polygone, welche zugleich einem kleinen Kreise der Kugel eingeschrieben und einem andern umgeschrieben sind." (*Journ. f. Math.*, 5, 1830); "Ueber die Anwendung einiger Formeln aus der Theorie der elliptischen Functionen auf ein bekanntes Problem der Geometrie" (*Ib.*, 38, 1849).

vestigation of the relation which must be satisfied by the coefficients of the two conics in order that there may be a polygon of $n$ sides inscribed in one and circumscribed about the other; generalising as needful the method invented by Jacobi and employing the elliptical functions of Jacobi.[1] That we can reach the same result using the analogous functions preferred by Weirstrass, was shown first by M. Simon[2] and afterwards by Halphen (1844–1889)[3]—in the second volume of his classic *Traité des fonctions elliptiques et de leurs applications* (Paris, 1888)—and by Vivanti.[4]

Having preferred in our exposition the logical order to the chronological, it is necessary to retrace our steps some decades to take note of the extremely elegant works of Nicola Trudi (1811–1884)[5] and those of A. Cayley (1821–1895)[6] not less interesting but much better known and even more widely diffused on account of being included in the collected papers of this celebrated scientist

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[1] The same analytical auxiliaries are employed in Rogers "Note on the Porism of the Inscribed and Circumscribed Polygon" (*Proc. L. M. S.*, 16, 1884–85).

[2] "De relationibus inter constantes duarum linearum secundi ordinis ut sit polygonum alteri inscriptum circumscriptum alteri" (*Diss.*, Berlin, 1867), and "Ganzzahlige Multiplication der elliptischen Functionen in Verbindung mit dem Schliessungsproblem" (*Journal für Math.*, 81, 1876). See Gundelfinger, "Ueber das Schliessungsproblem bei zwei Kegelschnitten" (*ibid.*, 83, 1877).

[3] C. Jordan, "Georges Halphen" (*Journal de Math.*, IV., 5, 1889); "Liste des travaux mathématiques de Georges-Henri Halphen" (*Palermo Rend.*, 3, 1889); Brioschi, "Notizie sulla vita e sulle opere di G. E. Halphen" (*Lincei Rend.*, IV., 5, 1889).

[4] "Sull' applicazione della funzione ellitica pu alla teoria dei poligoni di Poncelet" (*Palermo Rend.*, 7, 1893).

[5] *Sui poligoni iscritti e circoscritti alle curve coniche con date condizioni*, Naples, 1841; "Rappresentazione geometrica immediata dell' equazione fondamentale della teoria delle funzioni ellittiche con diverse applicazioni," *Naples Rend.*, 1843; "Studi intorno ad una singolare eliminazione con applicazione alla ricerca della relazione tra gli elementi di due coniche, l'una inscritta e l'altra circoscritta ad un poligono, ed ai corrispondenti teoremi di Poncelet" (*Napoli Atti*, 1863), and "Sui teoremi di Poncelet relativi ai poligoni iscritti e circoscritti alle coniche" (*Giorn. di Mat.*, 1, 1863).

[6] For biographical data concerning this geometer we refer the reader to the "Biographical Notice" by A. R. Forsyth which introduces Vol. VIII. (1895) of *The Collected Mathematical Papers of Arthur Cayley*; further to the "Notizia sulla vita e sulle opere di Arturo Cayley" by Brioschi (*Lincei Rend.*, V., 4, 1895), translated into French in *Bull. Sc. Math.*, II., 19, 1895; and lastly to the article "Arthur Cayley" by M. Nöther (*Math. Ann.*, 46, 1895).

(see Vols. 2, 3. 4, 5, 8, and 9); regarding these works we shall limit ourselves to note that from those of the English geometer we learn to write under the form of a determinant the simultaneous invariant of two ternary quadratic forms, whose vanishing announces that the conics represented by these admit of a polygon of Poncelet of the given order.

Analogous results were obtained by Mention in the "Essai sur la problème de Fuss" (*Petersb. Bull.*, 1, 1860) utilising an observation set forth by Tchébycheff (1821–1894)[1] in an article "Sur la série du problème de Fuss" (*Id., ib.*); by Puiseux (1820–1883) in the "Note sur les polygones qui sont à la fois inscrits dans un cercle et circonscrits à un autre cercle" (*Annales de la Soc. scient. de Bruxelles*, 3$^e$ ann., 1878–1879, 2$^e$ partie); by Kluyver in the memoir "Over de invariante betrekking tusschen twee kegelsneden in en om denzelfden velhoek beschreven" (*Nieuw Archief voor Wiskunde*, 15, 1888) and much earlier by Moutard in an article[2] which the originality of its concepts and the amplitude of its views places in a most eminent position among those referring to the theory which occupies us.

Lack of space forbids dwelling on the very numerous works whose modest aim was to demonstrate by methods different from those of Poncelet and Jacobi the results obtained by these, and even on the more significant ones which have for aim the study of the geometrical properties of the polygons of Poncelet.[3] But we wish to note how the theorem established by the great French geometer makes a part to-day of the splendid collection of *theorems of closure* (*Schliessungsprobleme*), to the constitution of which Steiner greatly contributed by discovering in 1832 one relative to circles on the plane or on the sphere (see *Systematische Entwicke-*

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[1] A. Vassillief, *P. L. Tchébycheff et son œuvre scientifique* (Loria Bollettino, 1, 1898).

[2] "Recherches analytiques sur les polygones simultanément inscrits et circonscrits à deux coniques," in the appendix to the first volume of the *Applications* of Poncelet already cited.

[3] See in particular: Halphen, "Application de la théorie des caractéristiques pour les coniques à une question relative aux polygones de Poncelet" (*Bull. Soc. Phil.*, VII., 3, 1876).

*lung* etc.[1]) and one still more noteworthy relative to polygons of an even number of sides inscribed in a plane cubic (*Journ. f. Math.*, 32, 1846), to demonstrate the latter of which many celebrated geometers exerted themselves, among which it is sufficient to mention Clebsch (*Journ. f. Math.*, 63, 1864), Ed. Weyr (*ib.*, 71 and 73, 1870–1871), Schoute (*ib.*, 95, 1883), Hurwitz (*Math. Ann.*, 19, 1882), Küpper (*ib.*, 24, 1884), Disteli (*Die Steiner'schen Schliessungs-probleme nach darstellend-geometrischer Methode*, Leipzig, 1889; see also Fiedler, *Die darstellende Geometrie*, etc., III. Th., p. 352 et seq., Leipzig, 1888), Martinetti (*Palermo Rend.*, 5, 1891), Czuber (*Journ. f. Math.*, 114, 1892). Some analogous propositions are due to August (*Archiv der Math.*, 59, 1876), to Harnack (1851–1888)[2] (*Math. Ann.*, 12, 1877), to Westphal (*Math. Ann.*, 13, 1878), to Forsyth (*Proc. L. M. S.*, 14), to Juel (*Nyt Tidss. f. Math.*, 1, 1890), to J. Thomae (*Leipziger Ber.*, 47, 1895), to R. A. Roberts (*Quart. Journ.*, 29, 1898), and to others whom for brevity we omit.

Instead we observe how the examination of the collection of works referring to the theorems of closure in general gives evidence of an intimate connection between these and the theory of elliptic functions. Now, just as it has been acutely noted[3] that in mathematics the appearance of any contradiction whatsoever indicates the presence of a hidden truth capable of adjusting the momentary discord, so we may say that a point of contact between two heterogeneous doctrines must find its justification in some more general truth. Thus for the theorems of closure, this superior reason consists in this, that every one of them gives rise to a homogeneous relation doubly quadratic between two series of binary variables, that is to a relation of the same form as that which subsists between two elliptic functions with the same argument; this is what Cayley demonstrated in the memoir "On the Porism of the In- and Circumscribed Polygon" and the (2, 2) "Correspondence of Points

[1] Comp. Vahlen, "Ueber Steiner'sche Kugelketten" (*Zeitschrift*, 41, 1896).

[2] A. Voss, "Zur Erinnerung an Axel Harnack" (*Math. Ann.*, 32, 1888).

[3] H. J. Stephen Smith, "On the Present State and Prospect of Some Branches of Pure Mathematics" (*Proc. L. M. S.*, 8, 1876, p. 25).

on a Conic" (*Quart. Journ.*, 11, 1871).[1] And these observations disclosing that part of the reasoning common to all the demonstrations of the theorems of closure, led Hurwitz to write a memoir[2] in which we know not whether more to admire the breadth of view or the perfection of form, and with which we shall end this digression, for in vain we should seek a more worthy close.

14. The papers of Poncelet "Mémoire sur les centres de moyennes harmoniques" (*Journ. f. Math.*, 3, 1828) and "Mémoire sur la théorie générale des polaires réciproques" (*ibid.*, 4, 1829), as well as the later "Analyse des transversales appliquée à la recherche des propriétés projectives des lignes et des surfaces" (*ibid.*, 8, 1832), make us approach the year 1837, in which was published the *Aperçu historique sur l'origine et le développement des méthodes en géométrie* by Michel Chasles (1793–1880)[3], a fascinating work in which the author, after having set forth in a style the beauty of which may be equalled but not excelled, all that constituted in his time the patrimony of pure geometry, vigorously defended the rights which this had to the consideration of scientists and which were continually questioned by the blind worshippers of analysis. One must not suppose however that this is a work of polemic alone, and that therefore it has, to-day that the strife is over, only a historical value. In fact from the *Mémoire de géométrie sur deux principes généraux de la science*, that follows the historico-critical part of the work in question, we learn the general properties of the collinear and reciprocal transformations whether in general or in the cases in which they are involutory, while in the thirty-four notes which accompany it are given important historical and scientific investigations. Of the first it is not necessary here to make explicit mention, and as to the others it will suffice to limit ourselves to

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[1] See also Chapters IX–X of the second part of the above-mentioned *Traité* of Halphen.

[2] "Ueber unendlich-vieldeutige geometrische Aufgaben, insbesondere über die Schliessungsprobleme" (*Math. Ann.*, 15, 1879); see also the note of the same geometer "Ueber die Anwendung der elliptischen Functionen auf ein Problem der Geometrie" (*ibid.*, 19, 1881).

[3] See J. Bertrand, "Éloge de Michel Chasles" in *Revue scientifique* of March 24th, 1892.

name those on cross ratio, on involution and on the law of continuity, since of the remainder, being more special, it is better to postpone the examination to a more opportune moment.

But in the interval of time interposed between the appearance of the writings of Poncelet and the publication of the work of Chasles, Germany had awakened from the torpor in which for half a century the soporific works of the "combinatorial school"[1] had immersed her,—a torpor from which the powerful voice of Gauss[2] (1777–1855), the *princips mathematicorum*, was not able to rescue her, he whom Jacobi did not hesitate to compare with Archimedes, finding in his works discoveries equally profound with those of the ancient, set forth in a form the most perfect, and with ideal scientific rigor; he whose mental preëminence is manifested, as Borchardt notes, in the decisive influence which he exercised on the mathematics of our time, penetrating in all his researches down to the very heart, clarifying and amplifying the fundamental concepts of mathematics, binding together under general laws facts which had remained unexplained or isolated, and uniting the rigor of the antique methods with the free play of modern analysis.

The reawakening in Germany of the spirit of investigation is marked by a new transference of the mathematical sceptre, which may be considered as having taken place in 1826, the year of the founding by the exertions of A. L. Crelle (1780–1855) of a periodical publication to which Abel (1802–1829)[3] and Jacobi, Steiner and Plücker, Moebius and von Staudt quickly insured renown not destined to decline; and it is precisely to these last four scientists that geometry is indebted for her return in honor to the east of the Rhine: to these it is therefore our duty to dedicate some lines of this introductory chapter.

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[1] The work done by this curious school who consider the polynomial theorem as the most important proposition of all analysis is analysed in Gerhardt, *Geschichte der Mathematik in Deutschland* (München, 1877) pp. 201–206.

[2] See Sartorius von Waltershausen, *Gauss zum Gedächtniss* (Leipzig, 1856); Schering, "Zur Feier der hundertsten Wiederkehr von Gauss' Geburtstag" (*Gött. Nachr.*, 1877).

[3] Bjerkness, *Niels-Henrik Adel. Tableau de sa vie et de son action scientifique* (Paris, 1885).

15. The most important of the works of Moebius (1790–1868)[1] in the field of geometry is that having for title *Der barycentrische Calcul* (Leipzig, 1827)[2]; herein the ancient ideas concerning the mass-center of a system of points are made the foundation of a most important algorithm, which leads to a new system of coördinates, of which the author demonstrates the applicability to the study of plane and twisted curves and to the research of the properties of new geometric transformations. Later applications of the same methods are contained in memoirs which the author inserted in Vols. 5 (1830), 24 (1842), 26 (1843), and 37 (1844) of the *Journ. f. Math.*, while new species of transformations (circular affinity, involution, symmetry, elementary transformations) were set forth by him in other writings, of which we do not cite the titles here in detail, nothing being easier than to find them now that the issue of his *Gesammelte Werke* has been completed. Others of his researches of a different kind will be mentioned later; here we make an exception only in favor of those on the signs of plane and solid figures (which among other things led to the ideas of polyhedra without volume and of unilateral surfaces), on the polyhedra, on the *Ausdehnungslehre* of H. Grassmann (1809–1877) and on the *Quaternions* of W. R. Hamilton (1805–1865), there being no point in our history more opportune for their mention.

Möbius contributed also to the progress of mechanics, of optics, of astronomy, as well as analysis, but these works are without the frame of our picture; if we mention them it is to indicate how they proclaimed a substantial difference between him and another of the stars of first magnitude which in that epoch illuminated the sky of German mathematics, J. Steiner (1796–1863)[3], who was so exclusively geometer that he was never willing to come to terms openly

[1] See the bio-bibliographic sketch prefixed by Baltzer to *August Ferdinand Möbius' gesammelte Werke* (Leipzig, 1885–1887).

[2] Note how a preceding work on "Zwei geometrische Aufgaben" (inserted in a volume of *Beobachtungen auf der K. Univ.-Sternwarte zu Leipzig*) shows that from 1823 Möbius knew and knew how to employ in a masterly manner the barycentric calculus.

[3] See C. F. Geiser, *Zur Erinnerung an Jacob Steiner* (Zurich, 1874); Graf, *Der Mathematiker Jacob Steiner von Utzendorf* (Bern, 1897).

with the analysts.[1] All know how in his *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander* (Berlin, 1832) "was revealed the organisation by virtue of which the most varied phenomena of space are connected among themselves," how here are rigorously established the principles by means of which that great geometer was able to accomplish the wondertul discoveries which will be separately cited in the course of our narrative; here we wish only to mention, in the first place how the possibility of founding on these a complete treatment of the conic sections appears from the *Vorlesungen über synthetische Geometrie* (Part I., 2nd ed., Leipzig, 1875; Part II., 3rd ed., *ibid.*, 1898) which two eminent disciples of Steiner (Geiser and Schröter) published under his name, and in the second place how he on several occasions demonstrated that pure reasoning, independent of calculation, can also be adapted to the measurement or the comparison of areas and volumes, and presented the most memorable example of this in the memoirs "On maxima and minima," to demonstrate the value of which it suffices to say that the calculus of variations was able to find only a long time after Steiner and by the way opened by him the means to follow synthesis in the solution of such questions.[2]

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[1] As is known, in the works of Steiner there are found innumerable propositions only enunciated; a complete catalogue of the writings devoted to their demonstration would prove extremely useful and is much to be desired.

[2] The original memoirs of Steiner on maxima and minima were completely published only after his death among his complete works; at first fragments of them had been published in the *Journ. f. Math.* and in the *Journ. de Math.* In this line of research Steiner was followed by Lindelöf ("Propriétés générales des polyhèdres qui, sous une étendue superficielle donnée, renferment le plus grand volume," *Math. Ann.*, 2, 1870, or *Petersb. Bull.*, 14, 1870; "Quelques problèmes relatifs à l'éllipse et à l'éllipsoïde," *Nouv. Ann.*, II., 10, 1871), by Berner Bauer ("Ueber Maxima and Minima geometrischer Figuren," *Zeitschr. f. Math.*, 11, 1866), by R. Sturm ("Bemerkungen und Zusätze zu Steiner's Aufsätzen über Maximum und Minimum," *Journ. f. Math.*, 96, 1884; "Würfel und reguläres Tetraeder als Maximum und Minimum," *ibid.*, 97, 1884), by Sturm and Lampe ("Ueber das Minimum des Inhaltes eines Vierecks bei gegebenen Seiten," *ibid.*), by Schwarz ("Beweis des Satzes, dass die Kugel kleinere Oberfläche besitzt als jeder andere Körper gleichen Volumens," *Götting. Nachr.*, 1884), and by E. Kötter ("Ueber diejenigen Polyeder die bei gegebener Gattung und gegebenem Volumen die kleinste Oberfläche besitzen," *Journ. f. Math.*, 110, 1892).

Less exclusive than Steiner was C. G. Von Staudt (1798–1867),[1] who did not completely separate himself from the influence of his great master Gauss and dedicated a part of his own intellectual activity to researches on the theory of numbers. A part, but not the most elevated and decisive, for this was absorbed by the solution of the great problem to treat the geometry of position without introducing any metrical concept; the *Geometrie der Lage* (Nürnberg, 1847), and the *Beiträge*, with which he adorned it in 1856, 1857, and 1860, certify to the complete result of his assiduous and genial efforts to rid of any extraneous element that projective geometry of which a little while before Poncelet and Chasles in France, Steiner and Möbius in Germany had laid the foundations. We cannot in few words delineate the content of these writings which justify for Von Staudt the enviable epithet of the "Euclid of the nineteenth century"; we shall only mention the new definition of projectivity there given, which contains only what is necessary and sufficient, and those of the conic and of the quadric, which, including also imaginary curves and surfaces, are able to rival the analytical definitions, and like these are susceptible of being extended to curves and surfaces of any order; we shall add that the researches on the imaginary in geometry[2] availed to put to flight the "spectre" which had pursued Steiner in the last years of his life, and to render superfluous the "principle of continuity" which had multiplied the opponents of the doctrine of Poncelet.[3] Other minor works of

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[1] See the study by C. Segre prefixed to *Geometria di posizione di C. G. von Staudt, trad. dal tedesco a curo del Dr. M. Pieri* (Torino, 1888).

[2] Comp. A. Ramorius, "Gli Elementi imaginari nelle Geometria" (*Giorn. di Mat.*, 35, 1897, e 36, 1898).

[3] Among all the investigations of Von Staudt these are undoubtedly the most abstruse; on this account their results spread with more difficulty in the mathematical world; to facilitate their comprehension, efforts were made in various directions by Lüroth (*Math. Annalen*, 8, 1875, and 11, 1877), August (*Programm der Friedrichs-Realschule*, Berlin, 1872), Stolz (*Math. Ann.*, 4, 1871); Henry J. Stephen Smith (*Ann. di Mat.*, II., 3, 1869–1870), H. Wiener (*Rein geometrische Theorie der Darstellung binärer Formen durch Punktgruppen auf den Geraden*, Darmstadt, 1885), Segre (*Torino Mem.*, II., 38, 1886, and *Journ. f. Math.*, 100, 1886) and Servais (*Belgique Mém.*, 49, 1896). To the theory of imaginaries of Von Staudt is allied the "Rechnung mit Würfen," to which Lüroth (*Mem. cited*), Sturm (*Math. Ann.*, 9, 1876), Schröder (*ibid.*, 10, 1876) and G. Kohn (*ibid.*, 46,

Von Staudt contain applications of the aforesaid doctrines, and, demonstrating how he knew duly to appreciate and in a masterly manner to treat metrical questions, make us regret that it was not given him to give to the *Geometrie der Lage* a sister work in the *Geometrie des Masses* which he had projected.[1]

A direction completely different from the investigations of Von Staudt have the publications of Julius Plücker (1801–1868),[2] to whom analytical geometry is indebted for decisive advancements; to him in fact we owe the development of homogeneous and polyhedral[3] coördinates, of the coördinates of the straight in the plane[4] and of the plane in space;[5] to him, omitting for a moment

1895) have devoted special researches. In this last work is extended the concept of *Würfe* to a group of $n$ elements of a form of I species; such extension is applied in the later work of the same geometer entitled "Die homogenen Coordinaten als Wurfcoordinaten" (*Wien Ber.*, 104, 1895). A theory of imaginary elements different from that of Von Staudt is created by A. Mouchot and expounded in the works: *La réforme Cartésienne étendue aux diverses branches des mathématiques* (Paris, 1876) and *Les nouvelles bases de la géométrie supérieure* (Paris, 1892).

[1] It is by inspiring themselves with the conceptions of Von Staudt, or rather by. unfolding them excellently, that Juel (in the dissertation "Bildrag til den imaginaere Linies og den imaginaere Planis Geometri," Kopenhagen, 1885, and in the article "Ueber einige Grundgebilde der projectiven Geometrie," *Acta*, 14, 1889) and Segre (in the interesting group of notes on "Un nuovo campo di ricerche geometriche," inserted in *Torino Atti*, t. 25 and 26, 1890 and 1891) discovered new correspondences and new figures which it is necessary to consider to exhaust completely the projective geometry of the plane (that is with real and complex points). In the same direction Sforza proceeded in writing the "Contributo alla geometria complessa" (*Giorn. di Mat.*, 30, 1892).

[2] See A. Clebsch, "Zum Gedächtniss an Julius Plücker" (*Götting. Abh.*, 15, 1872). Beltrami justly notes to this discourse that "the best eulogy which can be made of Plücker, considered as a geometer, is this, that Clebsch was not able to weave the account of his works, without recounting to a large extent the history of modern analytical geometry" (*Giorn. di Mat.*, 11, 1873, p. 153). Besides numerous memoirs published mostly in the *Journ. f. Math.*, we are indebted to Plücker for five great geometrical works; they are, *Analytisch-geometrische Entwickelungen* (Essen, 1828–1831), *System der analytischen Geometrie* (Berlin, 1835; see the *Anzeige* given of it by Plücker himself in *Journ. f. Math.*, 10, 1833), *Theorie der algebraischen Curven* (Bonn, 1839), *System der Geometrie des Raumes* (Düsseldorf, 1846), and *Neue Geometrie des Raumes* (Leipzig, 1868–1869).

[3] "Ueber ein neues Coordinatensystem" (*Journ. f. Math.*, 5, 1829).

[4] "Ueber eine neue Art, in der analytischen Geometrie Punkte und Curven durch Gleichungen darzustellen" (*ibid.*, 6, 1829).

[5] "Note sur une théorie générale et nouvelle des surfaces courbes" (*ibid.*, 9, 1831).

the geometry of the straight in space of which we shall treat *ex professo* later, we owe finally varied and most important applications of the "method of abridged notation"[1] of which he is one of the creators,[2] and of that of the "enumeration of constants" which, often but not always, he knew how to employ fitly.[3] If we here dispense with enumerating at this moment the new results for which our science is indebted to him, it is that we consider it much more convenient to do it in describing the successive evolution of the individual theories which constitute modern geometry, and to which it is now time for us to turn, having finished the sketch of the intellectual movement which prepared the present epoch. We thus shall see how the great men of whom we just now learned have been followed by a numerous and brilliant cohort of disciples, who, gleaning in the fields plowed by the masters, proved the fecundity of the seed which these had sowed.[4]

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[1] "Ueber ein neues Princip der Geometrie und den Gebrauch unbestimmter Symbole und Coefficienten" (*ibid.*, 5, 1829); "Analytisch-geometrische Aphorismen" (*ibid.*, 10 and 11, 1831); "Ueber Curven dritter Ordnung und analytische Beweisführung" (*ibid.*, 34, 1847).

[2] Of this procedure (invented also by Bobillier) Plücker indicates the most valuable qualities by the following words: "Meine Gleichungsformen sind vollständige Darstellungen graphischer Constructionen, in denen nichts Fremdartiges sich findet; es sind ideale, mit analytischen Symbolen hingezeichnete Figuren." (*Journ. f. Math.*, 34, 1847, p 332.)

[3] No one ignores how dangerous is this artifice, otherwise very fertile (see for this the recent memoir published by Küpper in the *Math. Ann.*, 32, 1888); Plücker, who knew and boasted of its qualities, knew moreover its drawbacks and often succeeded in avoiding them in the manner described by Clebsch in the afore-mentioned Commemoration.

[4] The foregoing *Sketch* may be followed by other extracts from Loria's work, giving the history of modern geometry as above indicated. But the editors are at present unable to promise how much will be offered to their readers.—*Ed. Monist.*